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ON THE FREQUENCY-DEPENDENT EFFECTIVE VISCOSITY OF LAMINAR AND TURBULENT FLOWS

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Abstract. The efficiency of tidal dissipation in convective zones of stars and giant planets depends, in part, on the response of a three-dimensional fluid flow to the periodic deformation due to the equilibrium tide — a problem considered by Jean-Paul Zahn in his PhD thesis. We review recent results on this problem and present novel calculations based on some idealized models.

1 Introduction

Astrophysical tides are important in close binary stars and short-planet exoplanetary systems, where they lead to evolution of the spin and orbital parameters on astronomical timescales. A major theoretical challenge is to quantify the rate of dissipation when a star or planet undergoes periodic deformation as a result of tidal forcing, as this determines the rate of evolution (*e.g.* Ogilvie 2014).

Jean-Paul Zahn's PhD thesis on the subject of tides in close binary stars was published in a series of three papers (Zahn 1966a, 1966b, 1966c). Among other achievements of this seminal work were the identification of *equilibrium* and *dynamical* tides (consisting respectively of a large-scale quasi-hydrostatic spheroidal bulge and small-scale internal waves) and an appreciation of the fact that convection is less efficient in damping tides when the convective timescale τ_c is long compared to the tidal oscillation period $2\pi/\omega$.

In Zahn (1966b) (see also Zahn 1989) he considered turbulent convection in the form of a cascade of eddies with sizes l and timescales $\tau \sim l/u$. If the distance $l' \sim \pi u/\omega$ through which an eddy turns in half an oscillation period is less than l, he argued that the eddy viscosity at scale l should be estimated as $l'u/3$ rather than $lu/3$. Applied to a Kolmogorov spectrum, this leads to the conclusion that the eddy viscosity of turbulent convection is dominated by the largest scales, and is reduced from that estimated by mixing-length theory by a factor of order $\pi/(\omega\tau_\mathrm{c})$, where this is less than unity.

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Goldreich & Nicholson (1977) argued instead that the eddy viscosity is dominated by scales at which the timescale $\tau \sim l/u$ is comparable to the oscillation period. For a Kolmogorov spectrum, this leads to a more severe reduction factor of order $1/(\omega\tau_{\rm c})^2$. Convective damping of the equilibrium tide is then much too weak to explain the circularization of solar-type binary stars or to have a significant effect on observed exoplanets around solar-type stars (*e.g.* Ogilvie 2014).

Penev *et al.* (2007) applied a formalism developed by Goodman & Oh (1997) to simulations of convection and deduced that the eddy viscosity scales as ω^{-1} at high frequencies, not because Zahn's theory was applicable but because the turbulent spectrum was much flatter than Kolmogorov. The perturbative formalism is questionable because it ascribes the damping to eddies with $\tau \sim 1/\omega$, having already assumed that $\omega\tau \gg 1$. Penev *et al.* (2009) found support for their conclusion from more direct attempts to measure the eddy viscosity.

Ogilvie & Lesur (2012) introduced the *oscillatory shearing box* for analytical and computational studies of flows subject to periodic deformation. They argued analytically that, in the limit that the oscillation period is short compared to all timescales in the flow, the dominant response is elastic, together with a weak effective viscosity that scales as ω^{-2} with a coefficient that could be positive, negative or zero. They found similar results from simulations of sheared turbulent convection, even when the tidal frequency was not so high.

2 The oscillatory shearing box

The oscillatory shearing box (OSB) is a convenient model for studying the response of a laminar or turbulent fluid flow to periodic deformation. Consider an incompressible fluid of unit density, satisfying the Navier–Stokes equations

$$(\partial_t + \boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla}p + \nu\nabla^2\boldsymbol{u} + \boldsymbol{f}, \qquad \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{u} = 0, \tag{2.1}$$

where $\boldsymbol{u}$ is the velocity, p is the pressure, ν is the kinematic viscosity and $\boldsymbol{f}$ is a body force. Suppose that the system is subject to a homogeneous time-dependent shear $a(t)$ in the xy plane, described by the coordinate transformation

$$x' = x, \qquad y' = y - a(t)x, \qquad z' = z, \qquad t' = t. \tag{2.2}$$

The Navier–Stokes equations written in these shearing coordinates (but using the original Cartesian basis vectors) are

$$[\partial_t' + \boldsymbol{v}\boldsymbol{\cdot}(\boldsymbol{\nabla}' - a\,\boldsymbol{e}_x\partial_y')]\boldsymbol{v} + \dot{a}v_x\,\boldsymbol{e}_y = -(\boldsymbol{\nabla}' - a\,\boldsymbol{e}_x\partial_y')p + \nu(\boldsymbol{\nabla}' - a\,\boldsymbol{e}_x\partial_y')^2\boldsymbol{v} + \boldsymbol{g}, \tag{2.3}$$

$$(\boldsymbol{\nabla}' - a\,\boldsymbol{e}_x\partial_y')\boldsymbol{\cdot}\boldsymbol{v} = 0, \tag{2.4}$$

where $\boldsymbol{v} = \boldsymbol{u} - \dot{a}x\,\boldsymbol{e}_y$ is the velocity relative to the shearing motion, and $\boldsymbol{g} = \boldsymbol{f} - \ddot{a}x\,\boldsymbol{e}_y$ is the body force that drives the internal flow in the box, rather than the large-scale shear. These equations are spatially homogeneous and are compatible with periodic boundary conditions. We work henceforth only in shearing coordinates and drop the primes. The (suitably averaged) Reynolds stress component

$$-R_{xy} = -\langle v_x v_y\rangle \tag{2.5}$$

quantifies the response of the internal flow to the large-scale periodic shear and determines the rate at which energy is exchanged between them. Our aim is to determine this rheology.

3 Linearized equations

Consider a basic flow $(\boldsymbol{v}, p)$ that exists in the absence of shear. The linearized equations in the presence of a small shear are

$$(\partial_t + \boldsymbol{v}\cdot\boldsymbol{\nabla} - \nu\nabla^2)\boldsymbol{w} + \boldsymbol{w}\cdot\boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\nabla}q = \boldsymbol{h}, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{w} = 0, \tag{3.1}$$

with

$$\boldsymbol{h} = -2\dot{a}v_x\,\boldsymbol{e}_y - 2a\nu\partial_x\partial_y\boldsymbol{v} + a\partial_y p\,\boldsymbol{e}_x - a(\partial_t + \boldsymbol{v}\cdot\boldsymbol{\nabla} - \nu\nabla^2)v_x\,\boldsymbol{e}_y, \tag{3.2}$$

where $\boldsymbol{w} + av_x\,\boldsymbol{e}_y$ is the velocity perturbation induced at first order by the shear, and q is the accompanying pressure perturbation. In determining the driving term $\boldsymbol{h}$, we have assumed that the body force $\boldsymbol{g}$ is fixed in shearing coordinates. The linearized Reynolds stress is

$$-\langle av_x^2 + v_xw_y + w_xv_y\rangle. \tag{3.3}$$

4 The ABC flow

The ABC flow (named after Arnol'd, Beltrami and Childress) is a widely studied family of exact solutions of the Navier–Stokes equations in a triply periodic cube (*e.g.* Galloway & Frisch 1987). It takes the form

$$\begin{aligned} \boldsymbol{v} &= (A\sin kz + C\cos ky, B\sin kx + A\cos kz, C\sin ky + B\cos kx), \\ p &= -AB\cos kz\sin kx - BC\cos kx\sin ky - CA\cos ky\sin kz, \end{aligned} \tag{4.1}$$

where k is the single wavenumber of the flow, while A, B and C are three independent amplitudes. It satisfies the steady Euler equations

$$\boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\nabla}p = 0, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{v} = 0 \tag{4.2}$$

and can be maintained against viscous dissipation by a body force $\boldsymbol{g} = \nu k^2\boldsymbol{v}$. We define the characteristic scales of length and time of the ABC flow by $L = 1/k$ and $T = L/U$, where $U = \sqrt{A^2 + B^2 + C^2}$ is the rms velocity. For sufficiently small Reynolds number $\mathrm{Re} = LU/\nu$, the ABC flow is stable. If the boundary conditions are periodic on a cube of length $2\pi L$, the critical Reynolds number is approximately 22.5 in the isotropic case $A = B = C$, and 14.3 in the sample anisotropic case $A = B = 2C$.

As the Reynolds number is increased, the ABC flow becomes irregular, develops smaller scales and eventually becomes turbulent (Podvigina & Pouquet 1994).

Braviner (2015) has used the OSB to study the response of the ABC flow to periodic shear in both laminar and turbulent regimes.

The laminar ABC flow has regions of quasiperiodic streamlines that are organized around six principal vortices, separated by chaotic regions (Dombre *et al.* 1986). The quasiperiods of streamlines starting at random points are broadly distributed up to about $20T$, with peaks close to $5T$ and $13T$ in the isotropic case.

5 Linear response of the laminar ABC flow

Given a sinusoidal shear $a \propto \cos(\omega t)$ (which in linear theory may be assumed arbitrarily to be of unit amplitude), the linear response is

$$\Re\left[(\tilde{\boldsymbol{w}}, \tilde{q})\,\mathrm{e}^{-\mathrm{i}\omega t}\right], \tag{5.1}$$

where $(\tilde{\boldsymbol{w}}, \tilde{q})$ satisfy

$$(\partial_t - \mathrm{i}\omega + \boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla} - \nu\nabla^2)\tilde{\boldsymbol{w}} + \tilde{\boldsymbol{w}}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\nabla}\hat{q} = \hat{\boldsymbol{h}}, \qquad \boldsymbol{\nabla}\boldsymbol{\cdot}\tilde{\boldsymbol{w}} = 0, \tag{5.2}$$

and (in the stable regime) tend to constant values as $t \to \infty$, with

$$\begin{aligned} \hat{h}_x &= CAk\sin kz\sin ky, \\ \hat{h}_y &= (2\mathrm{i}\omega - \nu k^2)A\sin kz - ABk\cos kz\cos kx, \\ \hat{h}_z &= 0, \\ \hat{q} &= \tilde{q} - (2\mathrm{i}\omega - \nu k^2)Ck^{-1}\sin ky + BC\sin kx\cos ky. \end{aligned} \tag{5.3}$$

(Here the irrotational part of $\boldsymbol{h}$ has been absorbed into $\boldsymbol{\nabla} q$.)

The linearized Reynolds stress is $\Re[G\,\mathrm{e}^{-\mathrm{i}\omega t}]$, where

$$G = G_\mathrm{r} + \mathrm{i}G_\mathrm{i} = -\langle v_x^2 + v_x w_y + w_x v_y\rangle \tag{5.4}$$

is the frequency-dependent complex effective elastic modulus of the ABC flow. G_r and G_i describe the components of the Reynolds stress that are in and out of phase with the shear. The effective elasticity and viscosity of the flow may be identified with G_r and $-G_\mathrm{i}/\omega$, respectively.

We solve Equation (5.2) numerically by a Fourier spectral method, evolved towards a steady-state response. Sample results are shown in Figures 1 and 2 for isotropic and anisotropic ABC flows, for a range of Reynolds numbers below the onset of instability. (We have confirmed a selection of these results using the fully nonlinear OSB model with a periodic shear of small amplitude.) In every case the effective viscosity is positive and decreases $\propto \omega^{-2}$ for $\omega T \gg 1$. The effective elasticity can be positive or negative. Both the elasticity and viscosity have finite limits as $\omega \to 0$, the latter being comparable to the mixing-length estimate of $LU/3$. At intermediate frequencies there is a non-trivial frequency-dependent response, which is more peaked for higher Re.

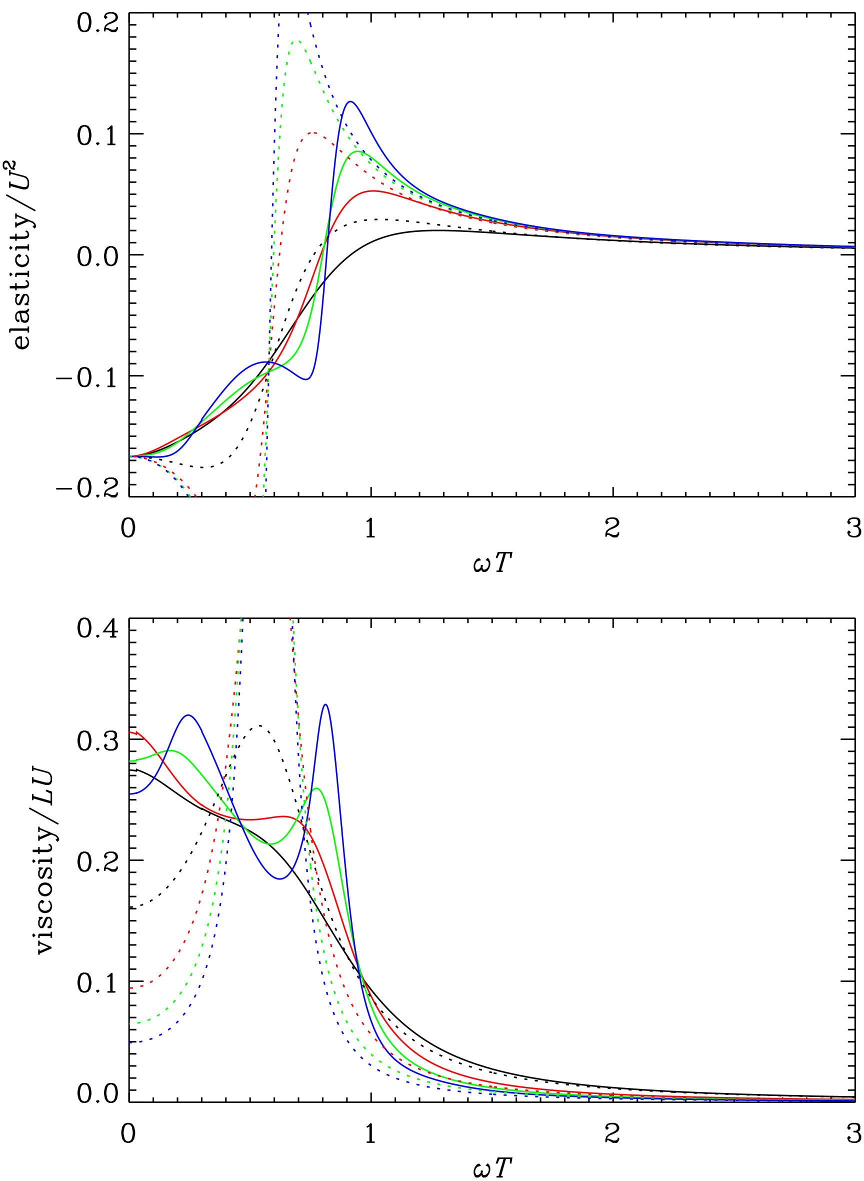
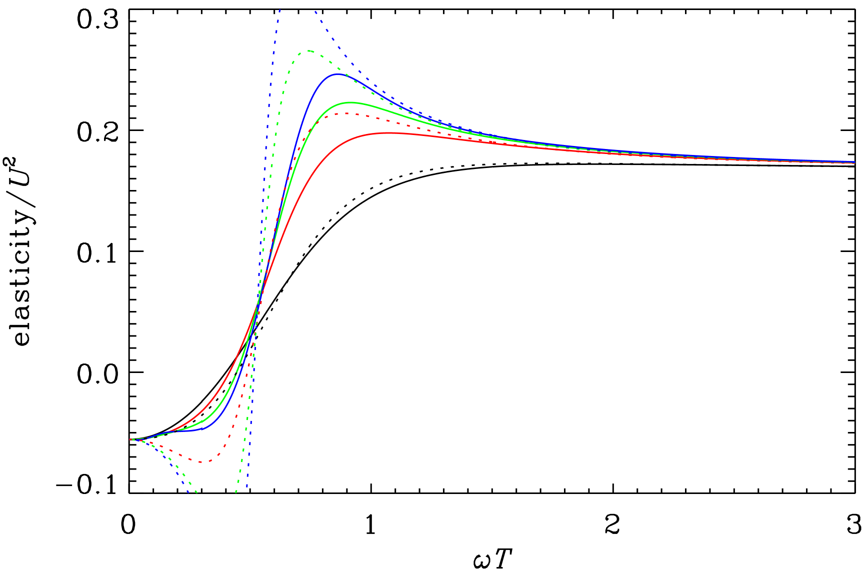


Fig. 1. Effective elasticity (*top*) and viscosity (*bottom*) of the isotropic laminar ABC flow with $A = B = C = 1$ *versus* frequency for Reynolds numbers of 5, 10, 15 and 20 (black, red, green and blue solid curves). The analytical approximation (5.6) is shown in the dotted curves of the corresponding colours.



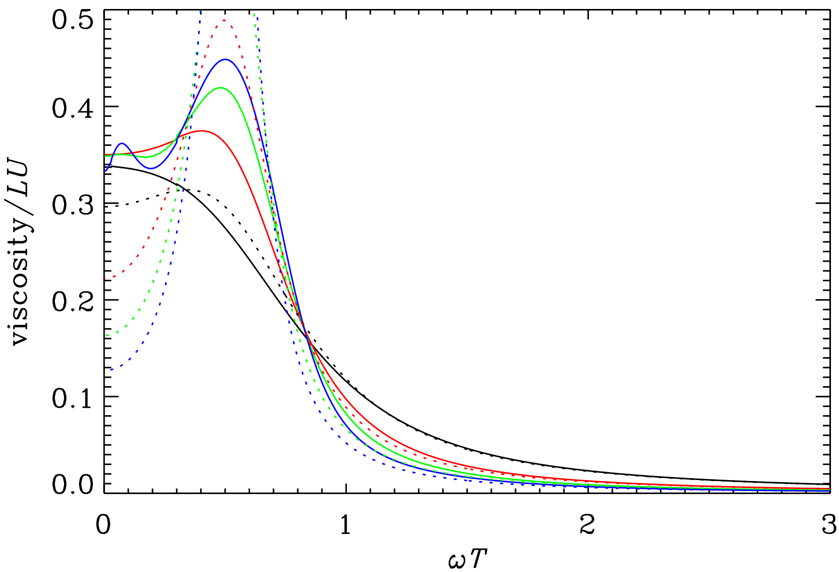


Fig. 2. Effective elasticity (*top*) and viscosity (*bottom*) of the anisotropic laminar ABC flow with $A = B = 2C$ *versus* frequency for Reynolds numbers of 3, 6, 9 and 12 (black, red, green and blue solid curves). The analytical approximation (5.6) is shown in the dotted curves of the corresponding colours.

Approximate analytical solutions can be obtained by a variety of methods. For example, if the equations are projected on to a truncated Fourier basis with wavenumbers $\leq \sqrt{2}k$, the solution involves a velocity perturbation of the form

$$\boldsymbol{w} \approx (c_1 \cos kz + c_3 \sin ky \sin kz, c_2 \sin kz + c_4 \cos kz \cos kx, 0) \tag{5.5}$$

and leads to a complex effective elastic modulus

$$G \approx \frac{1}{2}A^2 \left[\frac{\omega(\omega + 2\mathrm{i}\nu k^2)}{(\omega + \mathrm{i}\nu k^2)(\omega + 2\mathrm{i}\nu k^2) - \frac{1}{2}(B^2 + C^2)k^2} \right] - \frac{1}{2}C^2. \tag{5.6}$$

This can be further simplified in the limit of weak advection, in which either $\omega T \gg 1$ or $\mathrm{Re} \ll 1$, to

$$G \approx \frac{1}{2}A^2 \left(\frac{\omega}{\omega + \mathrm{i}\nu k^2} \right) - \frac{1}{2}C^2, \tag{5.7}$$

which is the combination of a constant negative elasticity with the response of a viscoelastic material. The more complicated expression (5.6) is plotted in Figures 1 and 2 for comparison; closer agreement can be obtained by using less radical truncations of the Fourier basis. The peaks at intermediate frequencies can plausibly be interpreted as a resonance with a stable oscillation mode of the ABC flow; they also correspond to oscillation periods comparable to the range of quasiperiods of ABC streamlines.

In the high-frequency limit, as found by Ogilvie & Lesur (2012), the elasticity tends to $\frac{1}{2}(A^2 - C^2)$ (which could be positive, negative or zero depending on the parameters chosen for the ABC flow), and the viscosity $\sim \frac{1}{2}A^2\nu k^2/\omega^2$. While the ω^{-2} scaling is generic (and different from that proposed by Zahn 1966b), the scaling with ν is also significant. Eddy viscosity requires irreversibility, which here relies on the laminar viscous dissipation of the ABC flow.

6 Turbulent ABC flows

Braviner (2015) has studied the response of a moderately turbulent ABC flow to periodic shear, using direct numerical simulations in the OSB model. He confirmed that the response in the high-frequency limit is predominantly elastic, with a weak effective viscosity $\propto \omega^{-2}$, consistent with the predictions of Ogilvie & Lesur (2012). This response is dominated by large scales in the box and can be attributed to the near-perfect freezing of the vorticity field in this limit. The behaviour of the effective viscosity at lower frequencies is not dissimilar to that of the laminar flows plotted in Figures 1 and 2. Indeed Braviner (2015) found that the time-averaged turbulent ABC flow resembles the laminar ABC flow on large scales, but with a smaller amplitude. This can be explained by noting that the large-scale flow breaks up into smaller-scale motions that provide an eddy viscosity, weakening the response to the ABC body force and bringing the large-scale flow to a moderate effective Reynolds number. This mechanism also provides the irreversibility required for the large-scale flow to exhibit an effective viscosity when subject to periodic shear.

7 Conclusion

Much more could be done to understand and quantify the response of laminar and turbulent flows, including turbulent convection, to periodic deformation, and to apply the findings to tidal dissipation in stars and planets. Numerical simulations of strongly turbulent flows subject to high-frequency shear are very demanding and require careful quantification of the errors.

The ABC flow can be considered as a simplified model of a single scale in a turbulent convective flow. Some features of its "viscoelastic" response curves, appropriately scaled as in Figures 1 and 2, can be supposed to apply to each scale in convection. The effective viscosity is similar to the mixing-length estimate at low frequencies, is quadratically suppressed at high frequencies and exhibits mild enhancement at intermediate frequencies.

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